Contexts for column addition and subtraction
Everyday settings, like enjoying freshly baked cookies, can foster proficiency and enhance students’ conceptual understanding.

By Jorge M. López Fernández and Aileen Velázquez Estrella

Tradition has it that the success of primary school mathematics education must be measured against the ability of students to correctly execute the standard arithmetic algorithms of elementary school: column addition and subtraction, column multiplication, and long division. To many educators, this is an inadequate benchmark, because a lot of interesting mathematics in primary school does not depend on the ability of students to successfully apply specific arithmetic algorithms. In fact, this belief constitutes a major point of contention in the “math wars” of today, in conflict with the back-to-basics philosophy advocated by some of the contenders in these wars (Schoenfeld 2004). According to Schoenfeld (p. 258), broadly stated, Americans visualize mathematics education in grades K–8 as basic arithmetic; algebra and geometry are to occur beyond middle school. It is no exaggeration to say that the inability of elementary school students to execute the four basic algorithms is taken as evidence of the failure of the present educational system. When standardized testing at the state level shows that students obtain progressively diminishing scores in their ability to add, subtract, multiply, and divide (McCarron 2004; Rivera Arguinzoni 2010), parents and teachers become concerned about the usefulness of the “new” math curriculum, which seems to bring problem solving to the foreground while failing to give students the arithmetic competencies needed to solve problems. Schoenfeld (2004) remarks that the “problem-solving approach” of the NCTM Standards-driven curriculum was never accompanied with a deep understanding of the nature of thinking and problem solving, and thus the “problem-solving movement was superficial.”

We describe what we regard as a middle ground for resolving this dilemma. We can teach the arithmetic algorithms for column addition and subtraction in a familiar problem-solving setting and in a way that fosters proficiency in implementing the algorithms as well as a better conceptual understanding of them.

Mathematics in Context (MiC) in Puerto Rico (MeCPR)

A well-known middle school mathematics curriculum for grades 5–8, the Mathematics in Context series was developed in the early 1990s under the leadership of Thomas A. Romberg, at the National Center for Research in Mathematical Sciences Education (NCRMSE) of the School of Education at the University of Wisconsin–Madison. The series was completed through a collaborative effort between NCRMSE and Freudenthal Institute (FI) at the University of Utrecht in The Netherlands and Encyclopedia Britannica Educational Corporation. The Mathematics in Context (MiC) curriculum is based on the fundamental tenet of Realistic Mathematics Education (RME), originally proposed by Hans Freudenthal, which asserts the natural ability of children to reason about mathematical content in informal contexts—such as baking, shopping, or play time. According to RME (and in keeping with Freudenthal’s famous “reinvention principle” [Gravemeijer and Terwel 2000, p. 786]), learning mathematics entails no more and no less than the reinvention of mathematical knowledge on the part of students under the guidance of adults (their teachers). Of course, this “reinvention” builds on the aforementioned informal mathematical knowledge of students.

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As expected, RME contextual textbooks can vary greatly. Some designs present units of the curriculum as parts of broad contextual themes, attempting to develop in a seamless, coherent, and integrated manner a plurality of curricular areas, down to specific competencies that students are expected to master. In somewhat of a piecemeal fashion, other designs develop many subthemes that focus on specific competencies that students are expected to master (the design used in MeCPR). To be sure, the difference is a matter of degree, and both approaches have their place in RME. Good examples of the first approach are the original MiC units developed in the University of Wisconsin MiC project. This approach, however, requires that middle school students possess a level of reading proficiency that is not customary in our present educational system. This situation, in and of itself, has been viewed as a deterrent to the adoption of MiC units in the United States and Puerto Rico.

In Puerto Rico, mathematics education has been customarily thought of along the more traditional lines of elementary, intermediate, and high school. The category of middle school as such has not received the same amount of attention, nor has it elicited the same kinds of concerns, as it has in the United States. To accommodate Puerto Rican elementary school students’ different levels of reading proficiency, MeCPR materials were developed using the second curricular design mentioned above. The Department of Education of Puerto Rico (DEPR) commissioned MeCPR to develop the “official” elementary school mathematics curriculum in the setting of RME.

Our didactic proposals for teaching the arithmetic algorithms of elementary school education are made within the setting of the K–6 MeCPR project. The development of the curricular materials was accomplished as a collaborative effort between the Department of Mathematics at the University of Puerto Rico–Río Piedras and the Fi of the University of Utrecht in The Netherlands. Koeno Gravemeijer directed the Dutch team, and (author) Jorge M. López Fernández directed the Puerto Rican team. Integrating arithmetic algorithms was done in keeping with Freudenthal’s principle of guided reinvention (Freudenthal 1991; Gravemeijer 1994) within the general framework of Realistic Mathematics Education (RME) (Treffers 1991, p. 11).

In actual operational terms, this means that the algorithms are presented in contexts familiar to students, capable of promoting exploration and the actual reinvention of the algorithms themselves by students. In this article, we discuss our approach to column addition and subtraction algorithms. We refer readers who are interested in instructional design within the framework of RME to Gravemeijer (1994) and Treffers (1991).

### Algorithms for column addition and subtraction

Opinions vary regarding how the column algorithms for addition and subtraction should be presented in the primary school curriculum. There are those who advocate that students invent the algorithms by themselves, with little guidance from the teacher (Kamii and Joseph 1988), and those who favor direct and explicit teaching of the algorithms with few motivational activities. Since algorithms are explicit “recipes” for carrying out certain procedures, students are prone to make mistakes in executing the algorithms as they strive to remember details of their implementation, especially in the absence of context. Educational research shows that students who are exposed too early to the column algorithms are prone to make serious errors regarding the order of magnitude of the

![Research shows that exposing students to column algorithms too early contributes to their confusion about units, tens, and hundreds as illustrated by the student work below.](image)
resulting digits (for instance, confusing units, tens, and hundreds). In other words, the errors in the application of the algorithms to numbers up to 100 can be traced back to a lack of understanding of the relationship between tens and ones. Students tend to treat the tens and the ones columns as if they were completely independent of each other. This leads to errors in the application of the rules for regrouping (see, for instance, Dominick and Kamii 1989). This error is especially evident when "subtracting smaller from larger," that is, when the subtrahend is larger than the minuend (see fig. 1). Furthermore, some children show limited conceptual understanding of addition and subtraction of integers when compared to children who go through the development and application of strategies for mental addition and subtraction. For instance, in Dominick and Kamii (1989), one finds an assortment of the types of errors that second graders with some exposure to the algorithm for column addition make in solving three-digit addition problems. Perhaps readers can determine the source of errors that lead students to produce eight different answers (9308, 1000, 989, 938, 906, 838, and 295) for the stated problem $7 + 52 + 186 = \Box$. Our experience is that it is not uncommon for students with practice in mental arithmetic strategies to rearrange the problem into something like the following: $5 + 50 + 190 = 45 + 200 = 245$.

At an early stage in designing our curricular materials, it became evident that we needed an appropriate context within which students would be able to perform such operations as grouping by tens and exchanging tens and ones. Adapting an original idea of Paul Cobb and Erna Yackel's from *A Contextual Investigation of Three-Digit Addition and Subtraction* (McClain, Cobb, and Bowers 1998) related to packing and unpacking candy in a candy factory, we provided an analogous context by designing activities concerning the packing and unpacking of cookies in a bakery setting. This context, proposed by the Dutch team (Gravemeijer, van Groenestijn, den Hertog, Carvalho de Figueiredo, and members of their support team at the FI), would allow students to construct the algorithms in a way that empowers them to make decisions related to figuring out what to do in particular problem situations as opposed to remembering what ought to be done. Observational experiences were carried out at the Carmen D. Ortiz school, and data were gathered regarding student understanding—both conceptual understanding of numbers and operations as well as understanding of the algorithms themselves. The remainder of this article describes in more detail the context employed; discusses how this context developed into the usual column algorithms for addition and subtraction; and finally, contrasts our approach to other approaches used in teaching place value and the algorithms for column addition and subtraction proposed in the literature. We hope this discussion will give interested readers an indication of how to reproduce the present activities and procedures in their own classrooms.

**Scope and sequence**

For reasons that are not always well understood, the discussion of place value and the presentation of the algorithms for column addition and subtraction occur early in American education. Early introduction of the column algorithms brings with it the danger that students will develop meaningless, error-prone routines. It is not uncommon to begin the study of place value in the first grade (with six-year-olds) and the discussion of the algorithms for column addition and subtraction in the second grade (with seven-year-olds). In sharp contrast, some European countries postpone teaching the column algorithms for addition and subtraction until children gain more sophistication in formal mathematics. For instance, in the
Netherlands, students invest a significant amount of time practicing arithmetic number strategies for mental arithmetic, which are directly applied to addition and subtraction. In reference to the traditional algorithms on column addition and subtraction, Treffers (1991, p. 48) states, “In grades 1, 2, and 3 there is no room for the standard algorithms.” He argues that mental arithmetic must be developed before students are able to handle the traditional column algorithms for addition and subtraction. Students are expected to learn the partitions of two-digit numbers that naturally arise in addition and subtraction problems. In doing so, and in solving addition and subtraction problems, children are expected to use the natural counting strategies developed early in primary school (such as counting by doubles, by triples, by fives, counting on or counting back from multiples of five or multiples of ten, etc.). For instance,

- $6 + 6$ is obtained by counting by doubles;
- $6 + 7 = (6 + 6) + 1 = 12 + 1 = 13$ is “almost” counting by doubles;
- $11 - 2 = 10 - 1 = 9$ is calculated by counting back from ten; and
- $8 + 5 = (8 + 2) + 3 = 10 + 3 = 13$ is calculated by counting on to ten.

Furthermore, in keeping with the strategy of developing mental arithmetic schemes for addition and subtraction (such as the ones mentioned), students are sometimes presented with problems in “column form” (in anticipation of the column algorithms) that require no-more-than or no-less-than rearrangements that can be treated with the discussed mental arithmetic techniques.

For instance, to solve the subtraction problem $23 - 17$, a student can subtract three (or add two) to each number to obtain the equivalent and friendlier subtraction problems:

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**Informal knowledge and nonstandard algorithms**

In solving column addition and subtraction problems of two or more digits, second graders are attracted to first handling the higher-order digits of the numbers being added or subtracted (Dominick and Kamii 1989). Children’s natural “attraction” toward the digits that they somehow regard as more important is a major obstacle for understanding the column algorithms. Such student behavior can be exploited to develop other algorithms for column addition and subtraction that differ significantly with the usual algorithms for column addition and subtraction. In solving the problem

A book has 64 pages. I have read 37 pages. How many pages are left to read?

Treffers (1991, p. 41) discusses the approach that students follow:
• Add 3 to get to 40;
• Add 20 more to get to 60; and
• Add 4 more to reach 64, for a total of
  \[2 + 20 + 4 = 27.\]

Since teachers generally regard this as a subtraction problem, some of them show amazement when students who “technically” cannot subtract are able to solve the problem simply by counting forward. In fact, children can employ the strategy of counting from 37 to 64 to get the answer (which, incidentally, is the way that people calculated change before the age of computerized cash registers). Treffers then discusses models that can be used to represent this approach to subtraction, among them what he calls empty number lines (i.e., non-calibrated number lines) to represent the “jumps” in moving on the number line from 37 to 64 (1991, p. 42). Interested readers might examine Mathematics in Context: Number Tools (1997, p. 12) to see how the strategies for “moving” on an empty number line are turned into the Jump, jump game, an intuitive model suggesting how to develop mental arithmetic while solving subtraction problems (see fig. 2).

**Cookie bakery context**

All activities designed for the two-digit column addition and subtraction for the second grade were placed in the setting of a cookie bakery run by a baker in charge of baking the cookies, and his wife, an accountant in charge of keeping the books. The bakery sells cookies individually, in packages of ten, and in boxes of ten packages. The idea is that the factory manager, Mrs. Bookkeeper, must be able to quickly take inventory of cookies at the bakery at any given time. In this context, all counting activities (especially adding and subtracting the number of cookies) are linked to activities of packing and unpacking cookies. Students normally begin packing and unpacking activities using real cookies, snap cubes, or other manipulatives (phase 1: concrete). They then proceed to making diagrams or drawings to represent problem situations in the bakery (phase 2: semi-concrete). Finally, students move to numerals to represent the number of cookies or number of packages in a given problem situation (phase 3: abstract). This progressive symbolizing and modeling of the packing of cookies within the cookie bakery context is tied to the decomposition of arithmetical units of different orders of magnitude (initially units and tens). Familiarity with such decompositions is certainly needed for the correct implementation of the column addition and subtraction algorithms. Below is a sampling of typical questions that can arise in this context. The questions have to do with the different ways to pack a certain number of cookies. Expected answers will depend on the level of abstraction (the phase) that students have reached.

- If the bakery sells approximately eight packages of cookies daily, how many do they sell altogether?
- **Keyla** has five packages of ten cookies, and there are seven left. If she wants to complete eight packages, show how many cookies are needed.
- **One** morning, Mr. Baker discovers that three packages and twenty-three unpacked cookies are left at the bakery from the previous day. How many cookies must he bake to complete eight packages?
- A **customer** wants to buy three packages of cookies just before closing time. Mr. Baker does not know if there are sufficient cookies in the store. He notices a package of cookies and seven unpacked cookies on the counter. If there are fifteen freshly baked cookies just out of the oven, can he sell the three packages of cookies that the customer wants?

The final stage in this process of progressive mathematization and symbolization is, of course, implementing the column algorithms. Certainly one of the most outstanding virtues of the context is that a “table scheme”—originally designed for the convenience of Mrs. Bookkeeper in her chore of keeping inventories of the cookies in the bakery—is transformed progressively into the usual column algorithms. Thus, the bakery scenario offers students all the added understanding that comes from the operations elicited by the model, such as the arithmetic decompositions that must be negotiated for keeping the bakery inventories. **Figure 3** shows the sheets designed for keeping inventories, one for addition and one for subtraction. Note that the left-hand columns are used to indicate the number of packages of ten cookies, and the right-hand columns, for indicating the number...
of unpacked cookies. Clearly one can make inventories in the bakery “warehouse,” keeping track of the additions (which originate from the baker sending newly baked cookies) and subtractions (which arise from sending new cookies to display in the cookie store). Also clear is that those people who are in charge of the store display can make their own inventories.

Students are enticed to solve such inventory problems while teachers progressively “decontextualize” the problem situations and do away with the inventory tables to get the conventional notation for column addition or subtraction. Students remember the cookie bakery metaphor long after they study the algorithms; the context is excellent for empowering students to correctly solve column addition and subtraction problems. The context allows students to solve problems related to the exchange of units and tens while adding or subtracting, and at the same time allows them to recognize the common errors related to the confusion of digits of different orders of magnitude.

Guided reinvention

The cookie bakery context allows students to invent and apply the column algorithms for addition and subtraction. The process described allows them to go from contextual situations related to cookie packaging to abstract situations having to do with the operations associated with column addition and subtraction, especially with regrouping for addition and subtraction. This process provides an example of how the principle of guided reinvention works within RME.

Typically, students are presented with familiar contextual situations to which they can relate through the informal mathematical knowledge that they bring with them to school. In our case, the context of the cookie bakery allows students to end up learning about exchanges between digits representing units and tens (and can be extended to hundreds for third graders) while trying to understand some issues that arise in a bakery. The context helps students avoid the mistakes associated with the rules for regrouping in addition and subtraction. It also averts the common mistake of “subtracting smaller from larger” when the subtrahend is larger than the minuend. According to the RME teaching methods, students actively engage in the design of “models” (such as the cookie bakery inventory sheets) to describe contextual situations, while negotiating issues having to do with packing and unpacking cookies in the bakery.

This process yields a transition from using actual cookies and snap blocks to symbolizing cookie packages by means of drawings or as digits on an inventory sheet (see fig. 4). Thus, students typically begin using descriptive models to understand problem situations in the bakery context. These models turn progressively more abstract and end up being prospective models.
capable of generating questions of mathematical value not envisioned in the original contextual situation.

Since a context underlies the use of the descriptive model as it gives rise to the prospective model, students master the use of the latter (in our case, column addition and subtraction), even after it has been stripped of its original context—a useful skill indeed. For instance, performing a cookie operation that requires subtracting a smaller minuend from a subtrahend is inconceivable; it just does not have any reasonable interpretation in the context of the cookie bakery. In the parlance of RME, the process of designing the descriptive model is horizontal mathematization, and the transition from the descriptive model to the prospective model is vertical mathematization (see fig. 5). Of course, RME instructional design capitalizes on this interaction between contexts and students’ capacity for symbolization to give reality to Freudenthal’s famous principle of guided reinvention. We refer interested readers to The Role of Contexts and Models in the Development of Mathematical Strategies and Procedures (Beishuizen, Gravemeijer, and van Lieshout 1997).
Conclusion
Introducing familiar contexts to elementary school students can be useful in promoting understanding of the standard arithmetic algorithms of primary school education. The context described illustrates how the principle of guided reinvention can be articulated within the framework of RME to improve students’ understanding—both at a conceptual level as well as at an operational level—when implementing the traditional column algorithms for addition and subtraction. By learning the algorithms in this fashion, students are able to uncover a wide collection of abstract questions related to the exchange of units and tens and the use of the algorithms. Even out of context, the cookie bakery imagery, with its implicit operations and actions, somehow remains with the students.

BIBLIOGRAPHY


